

Braneworld setup and embedding in teleparallel gravity

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Abstract

We construct the setup of a five-dimensional braneworld scenario in teleparallel gravity. Both cases of Minkowski and Friedmann-Robertson-Walker branes embedded in Anti de Sitter bulk are studied and the effective 4-D action were studied. 4-dimensional local Lorentz invariance is found to be recovered in both cases. However, due to different junction conditions, the equations governing the 4-D cosmological evolution differ from general relativistic case. Using the results of Ref. [13], we consider a simple inflationary scenario in this setup. The inflation parameters are found to be modified compared to general relativistic case.

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1. introduction

A few years after the introduction of general relativity (GR), Einstein proposed another gravitational theory called teleparallel gravity (TG) in an attempt to unify gravity with electromagnetism [1]. In this theory he considered a spacetime with zero curvature but nonzero torsion usually called teleparallel or Weitzenböck spacetime. This property was in contrast to general relativity in which the Riemannian spacetime had curvature with vanishing torsion. Einstein was not able to achieve his primary aim, as teleparallel gravity was not able to give the unification of forces. Moreover, the general teleparallel theory also was not invariant under local Lorentz transformations [2, 3]. By insisting on restoring the local Lorentz invariance, the theory becomes equivalent to general relativity, with empirically indistinguishable results, therefore it is called teleparallel gravity equivalent to general relativity (TEGR) [4, 5]. Nowadays, TG and GR are considered to be special cases of the more general gauge theory of gravity called Poincaré gauge theory (PGT) in which both torsion and curvature are present. In PGT the spacetime has a Riemann-Cartan structure and both the mass and the spin of the matter act as the sources of gravitational interaction. It offers the most realistic and satisfying theory of gravity [6].

On the other hand the attempt to overcome the problems in GR led to the born of variety of different proposals, see for example [7, 8] for reviews. A class of such theories modifies gravity by considering higher dimensions. The origin of the idea is a work by Kaluza and Klein [9] in which only one compact extra dimension has been considered. Through this assumption, they could unify gravity and electromagnetism. More recently, works by Randall and Sundrum led to a simple viable and cosmologically satisfying braneworld model. Their second model, RS II, has a single brane embedded in an infinite bulk. Although in the RS II setup, the extra dimension is not compact, the 4-D gravity is recovered on the brane [10]. Effectively in this setup the gravity is localized on the brane by the bulk curvature. The success of this model has attracted worldwide attention to extra dimensions and many researches and developments have been so far done in this area [11, 12].

Through this paper, we try to revive this five dimensional model in the context of a 5D TEGR background with the hope of finding probable intuitive differences between the two gravitational theories of TEGR and GR. The reason that we choose RS II as the framework is the existence of the

brane as a boundary surface which separates the two regions of the bulk. This gives an opportunity to analyze possible difference in the effective 4D gravity as the junction conditions which connect the 5D quantities to the stress-energy tensor of the brane, are found to be different in teleparallel gravity as shown in [13]. Another important question is that whether 4D local Lorentz invariance is recovered on the brane. This is not a straightforward question in teleparallel gravity as symmetric and gauge properties of this theory are fundamentally different from general relativity. 5D models based on teleparallel gravity are also studied in Refs. [14, 15].

The structure of the paper is as follows: after introducing notations and basic definitions in section II, we consider the case of a Minkowski brane embedded in an Anti de Sitter bulk in section III. This simple construction allows us to study the linearized gravity and weak-field limit on the brane effectively. For a more realistic braneworld setup, we study the case of a Friedmann-Robertson-Walker brane in a AdS bulk in section IV.

2. Notation and definitions

Throughout this paper the capital middle Latin letters M, N, \dots run over 0, 1, 2, 3, 5 and label spacetime coordinates. Lower case Latin letters from the beginning of the alphabets a, b, \dots run over 0, 1, 2, 3, 5 and label tangent space coordinates. The Greek indices μ, ν, \dots run over 0, 1, 2, 3 and refer to the 4D spacetime coordinates. Finally lower case Latin letters i, j, \dots run over 0, 1, 2, 3 and refer to the 4D tangent space coordinates.

In teleparallel gravity one considers a set of (pseudo)-orthogonal D-vectors (D is the number of spacetime's dimensions) which form a basis in the tangent space on every point of the manifold $\mathbf{e}_i \cdot \mathbf{e}_j = \eta_{ij}$. These bases are called tetrads in four dimensions (or pentads in 5D) and relate the manifold and Minkowski metrics through the relation

$$g_{\mu\nu} = \eta_{ij} e_\mu^i e_\nu^j \quad (1)$$

The inverse of the tetrad is defined by the relation $e_i^\mu e_\mu^j = \delta_i^j$. Here tetrads are the dynamical variables of the theory. InTEGR, the spin connection which determines the rule of the parallel transport in the tangent space, is assumed to be zero. The vanishing spin connection means that there is no spinor field in the theory. The weitzenböck connection is defined as [16]

$$\Gamma^\rho{}_{\mu\nu} = e_i^\rho \partial_\nu e_\mu^i \quad (2)$$

which unlike Livi-civita connection is not symmetric on its second and third indices. Curvature can be defined with respect to spin connection and torsion with respect to vierbeins and spin connection. Since the spin connection in this theory is zero, curvature is also turned out to be zero and then the torsion tensor is

$$T^\rho{}_{\mu\nu} \equiv e_i^\rho (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i). \quad (3)$$

Contorsion tensor which denotes the difference between Livi-civita and Weitzenböck connections is

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}) \quad (4)$$

and the superpotential tensor is defined as

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\alpha\nu}{}_\alpha - \delta_\rho^\nu T^{\alpha\mu}{}_\alpha). \quad (5)$$

In correspondence with Ricci scalar, one can define torsion scalar

$$T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} \quad (6)$$

The gravitational action in TEGR is

$$I = \frac{1}{16\pi G} \int dx |e| T \quad (7)$$

where $|e|$ is the determinant of e_μ^a and from the relation (1), one can easily finds that it is equal to $\sqrt{-g}$. Variation of the above action with respect to vierbeins gives the field equations in TEGR

$$e^{-1} \partial_\mu (e e_i^\rho S_\rho{}^{\mu\nu}) - e_i^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} + \frac{1}{4} e_i^\nu T = 4\pi G e_i^\rho \Xi_\rho{}^\nu \quad (8)$$

where $\Xi_\rho{}^\nu$ is the energy-momentum tensor.

The Lagrangian of TEGR (Eq. (7)) is consisted of just the torsion scalar T . Torsion scalar differs only by a total divergence term from Ricci scalar R of general relativity, $R = -T - 2\nabla_\mu T^\mu{}_\nu$. R is local Lorentz invariant but the total divergence term $\nabla_\mu T^\mu{}_\nu$ is not, which means that contrary to R , the torsion scalar cannot be local Lorentz invariant. The ∇ operator here is with respect to the Levi-civita connection. This total divergence term will vanish in an action integral and the field equations will become equal to that of GR. In five dimensional TEGR with RS II as the framework, to see if the effective theory also remains invariant under local Lorentz transformations, one should

study the behavior of transformed quantities when one projects them on the brane. Under such transformations in the tangent space, vierbein and torsion tensor transform as

$$e_M^a \mapsto \Lambda_b^a e_M^b \quad (9)$$

$$T_{NQ}^M \mapsto T_{NQ}^M + \Lambda_a^b e_b^M (e_Q^c \partial_N \Lambda_c^a - e_N^c \partial_Q \Lambda_c^a) \quad (10)$$

therefore the total divergence term in the five dimensional Lagrangian becomes

$$\nabla_N T_M^{NM} \mapsto \nabla_N T_M^{NM} + \nabla_N (\Lambda_a^c \partial^N \Lambda_c^a - \Lambda_a^b e_b^N e_M^c \partial^M \Lambda_c^a) \quad (11)$$

It is clear that the term which involves the Lorentz transformation tensor is completely separated from the untransformed torsion tensor and is still a total divergence. To obtain the effective Lagrangian on the brane, one should integrate the 5-D Lagrangian with respect to the extra dimension y . If the integration of the second term with respect to y vanishes, the effective Lagrangian on the brane remains invariant. This depends on the specific geometry of the bulk and the brane and we will examine this for the cases of Minkowski and FRW branes in the following sections. However, since the transformed part is still a total divergence term, when integrating over whole spacetime, it will definitely vanishes and leaves the 4-D effective field equations invariant independent of geometry. To sum up, although in general, the 4-D effective Lagrangian in TEGR is not equivalent with that of GR due to different junction conditions, but the theory remains local Lorentz invariant on the level of field equations.

Here we briefly present the main results of Ref [13]. In that paper, starting from a 5D Randall-Sundrum setup, the effective 4D field equations on the brane were derived by projecting all the 5D geometrical quantities using the equivalent of Gauss-Codacci equations in teleparallel gravity. The energy-momentum tensor can be considered as

$$\Xi_{MN} = -\Lambda_5 g_{MN} + \delta(y) \Omega_{MN} \quad (12)$$

where Λ_5 is the five dimensional cosmological constant, λ is the brane tension and Ω_{MN} is the matter stress-energy tensor of the brane. To find the induced field equations on the brane, using the procedure first introduced in [17], we project all the five dimensional quantities by using the projection pentad [13]

$$h_a^M = e_a^M - n^M n_a. \quad (13)$$

this projection tensor when acts on a vector, will project it on the brane and turns the tangent indices into coordinate and vice versa.

In RS II, brane is actually a border which divides bulk into two regions. Going from one side of the brane to the other, will cause some discontinuities in our physical quantities. To encounter these discontinuities one needs junction conditions. Expressing the pentad field as a distribution and requiring that the connection also be a distribution, we reach the first junction condition

$$[e_M^a] = 0 \quad (14)$$

This means that the pentad is continuous across the brane. Expressing other geometrical quantities in the same way and using the five dimensional field equations, we reach the second junction condition which guaranties the geometry of the theory remains well-defined. This relates the jump of the five dimensional superpotential tensor across the brane, to the matter content on the brane as [13]

$$e_a^O [S_O^{MN}] n_M = 4\pi G \Omega_a^N \quad (15)$$

Substituting projected quantities on the brane and using the above junction conditions and imposing the Z_2 -symmetry, we obtain the induced field equations on the brane [13]

$${}^{(4)}F_a^N = -\Lambda_5 h_a^N + (4\pi G_5)^2 \Pi_a^N + E_a^N \quad (16)$$

where we have defined

$$\begin{aligned} \Pi_a^N = & -\frac{3}{4} h_O^b \Omega_b^N \Omega_a^O + \frac{3}{8} h_a^O \Omega \Omega_O^N + \frac{1}{32} h_a^N \Omega_b^O \Omega_O^b + \frac{1}{32} h_a^N \Omega^2 \\ & + \frac{1}{4} \Phi^2 (1 + L_M J^M) \delta_O^N \Omega \Omega_a^O + \frac{1}{4} \Phi^2 (1 + L_M J^M) \delta_a^N \Omega^2 \end{aligned} \quad (17)$$

and

$$\begin{aligned} E_a^N = & n^O n_a \partial_M (S_O^{MN}) + S_O^{MN} (n^O \partial_M n_a) + S_O^{MN} (n_a \partial_M n^O) + h_a^O S_O^{MN} (n^b \partial_M n_b) \\ & + \left[n^M n_O n_b n^c e_d^N + n^N n_O n_d n^c e_b^M + n^M n_b n_d n^N e_O^c \right. \\ & \left. - n^c n_O e_b^M e_d^N - n^M n_b e_O^c e_d^N - n^M n_O n_b n^c n_d^N \right] S_c^{bd} \partial_M (h h_a^O) \end{aligned} \quad (18)$$

where pentad and the inverse pentad are given by

$$e_\mu^i(x, y) = \begin{pmatrix} e_\alpha^a & 0 \\ e_\alpha^{.5} & e_5^{.5} \end{pmatrix}$$

$$e_i^\mu(x, y) = \begin{pmatrix} e_a^\alpha & e_{.5}^\alpha \\ 0 & e_{.5}^5 \end{pmatrix}$$

respectively and we have defined

$$e_\mu^{.5} = L_\mu \Phi \quad e_5^{.5} = \Phi \quad , \quad e_i^5 = -h_i^\mu L_\mu \quad e_{.5}^5 = \Phi^{-1} \quad (19)$$

and $J_i = \Phi^{-1} \partial_i \Phi$. A ‘.’ in front of an index refers that it is a tangent space index.

3. Setup of the Randall-Sundrum model inTEGR

3.1. Five dimensional geometry setup inTEGR

In this section we wish to investigate a RS type scenario in the context of teleparallel gravity. Here the gravitational interactions are described by torsion instead of curvature. One of the fundamental assumptions of the original RS model was that the metric ansatz obeys the 4-D Poincare invariance.

Anti de Sitter space is the maximally symmetric solution of Einstein’s equations with an attractive cosmological constant. It has constant negative scalar curvature. The AdS_5 metric usually takes the form

$$\begin{aligned} ds^2 &= dy^2 + e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu \\ &= dy^2 + e^{2A(y)} [dx^2 + dy^2 + dz^2 - dt^2] \end{aligned} \quad (20)$$

This will be a AdS_5 metric if we set $A = \pm b$. $K = -b^2$ is the constant negative curvature of AdS_5 space in general relativity. By a simple coordinate transformation in the form of $v = \frac{e^{\mp by}}{b}$ the above metric can be transformed into:

$$ds^2 = \frac{1}{b^2 v^2} [dv^2 + dx^2 + dy^2 + dz^2 - dt^2] \quad (21)$$

we can see that the AdS_5 metric is indeed conformally flat as

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \quad (22)$$

where

$$\Omega^2 = e^{\pm 2by} \quad (23)$$

This coordinate system is usually called the 'stereographic coordinates' in the literature. In Teleparallel gravity we have [18]

$$g_{\mu\nu} = e^i{}_\mu e^j{}_\nu \eta_{ij} \quad (24)$$

where $e^i{}_\mu$ is the tetrad field. By using equation (22) and (24) we see that the tetrad field for the AdS_5 space in the stereographic coordinate is [18]

$$e^i{}_\mu = \Omega \delta^i_\mu \quad (25)$$

Weitzenbock connection has the form

$$\Gamma^\rho{}_{\mu\nu} = e_i{}^\rho \partial_\mu e^i{}_\nu \quad (26)$$

Using the AdS_5 tetrad (6) we have

$$\Gamma^\rho{}_{\mu\nu} = \delta^\rho_\nu \partial_\mu \ln \Omega \quad (27)$$

and the torsion will be

$$T^\rho{}_{\mu\nu} = \delta^\rho_\nu \partial_\mu \ln \Omega - \delta^\rho_\mu \partial_\nu \ln \Omega \quad (28)$$

So in this particular coordinate system the torsion has this simple form. Substituting for Ω from (23) shows us that the AdS_5 space in teleparallel gravity has constant negative torsion scalar and its curvature is identically zero. We can also easily show that the AdS_5 space is indeed the solution of the teleparallel field equation (8) with a negative cosmological constant where

$$\Xi_a{}^N = -\Lambda_5 e_a^N \quad (29)$$

By introducing the brane into this setup some restrictions will be imposed in the form of the warp factor. In order to have a well defined and acceptable Cauchy development, the warp factor in the presence of the brane should be

$$e^{2A(y)} = e^{-2b|y|} \quad (30)$$

This space is usually called the AdS_5/Z_2 space and is a choice which is bounded everywhere and unlike the unbounded $e^{2A(y)} = e^{2b|y|}$ it has a well developed Cauchy problem.

Using this warp factor and turning the attention to the original AdS_5 line element (20), one can see that the simplest pentad field in this setup is given by

$$e^a_M = \text{diag}\left(e^{-b|y|}, e^{-b|y|}, e^{-b|y|}, e^{-b|y|}, 1\right) \quad (31)$$

We now proceed to calculate the torsion and superpotential with this tetrad. We note that

$$\frac{d|y|}{dy} = \Theta(y) - \Theta(-y) = \epsilon(y) \quad , \quad \frac{d^2|y|}{dy^2} = 2\delta(y) \quad (32)$$

where $\theta(y)$ is the Heaviside distribution which is defined as follows: it is equal to +1 if $y > 0$, 0 if $y < 0$ and indeterminate if $y = 0$. It has the following properties

$$\Theta^2(y) = \Theta(y) \quad , \quad \Theta(y)\Theta(-y) = 0 \quad , \quad \frac{d}{dy}\Theta(y) = \delta(y) \quad (33)$$

where $\delta(y)$ is the Dirac distribution.

The non-zero components of the torsion are

$$T^0_{50} = -b(\Theta(y) - \Theta(-y)) \quad , \quad T^1_{51} = T^2_{52} = T^3_{53} = -b(\Theta(y) - \Theta(-y)) \quad (34)$$

and the non-zero components of the superpotential are

$$S_0^{50} = S_1^{51} = S_2^{52} = S_3^{53} = \frac{3}{2}b(\Theta(y) - \Theta(-y)) \quad (35)$$

So the torsion scalar will be

$$\begin{aligned} T &= -6b^2 \left(\Theta(y) - \Theta(-y) \right)^2 = -6b^2 \left(\Theta^2(y) + \Theta^2(-y) - 2\Theta(y)\Theta(-y) \right) \\ &= -6b^2 \left(\Theta(y) - \Theta(-y) \right) = -6b^2 \end{aligned} \quad (36)$$

So the AdS_5 space in teleparallel gravity indeed has constant negative scalar torsion.

If we denote the left hand side of the teleparallel field equation (8) by F_a^N , then we have

$$\begin{aligned} F_{.0}^0 &= e^{-3b|y|} 6b^2 - e^{-b|y|} 6b \delta(y) \\ F_{.1}^1 &= F_{.2}^2 = F_{.3}^3 = e^{-b|y|} 6b \delta(y) - e^{-3b|y|} 6b^2 \quad , \quad F_{.5}^5 = -e^{-b|y|} 6b^2 \end{aligned} \quad (37)$$

where a ‘.’ denotes the tangent space indices. From these equations we see that as well as having a cosmological constant in the bulk, there should be an additional energy momentum tensor on the brane (with a delta function). The complete energy - momentum tensor which supports this particular form of tetrad with this specific warp factor then should be

$$\Xi_a^N = -\Lambda_5 e_a^N + \lambda e_a^N \delta(y) \quad (38)$$

where λ is the cosmological constant induced on the brane or the brane tension. Using the teleparallel field equations we get

$$\lambda = \frac{6b}{\kappa_5^2} \quad , \quad \Lambda_5 = \frac{-6b^2}{\kappa_5^2} \quad (39)$$

So we also have

$$\kappa_5^2 \lambda^2 + 6\Lambda_5 = 0 \quad (40)$$

The presence of this additional energy-momentum tensor entails the presence of a new matter field λ which is localized to the $y = 0$ region which is associated with the brane localized there. In summary in this section we have constructed a 5-D AdS geometry in teleparallel gravity. Introducing the brane in this setup will induce some restrictions on the coefficient of the AdS pentad.

3.2. Effective 4-D action

The 5-D gravitational action in the RS setup is

$$S_{grav} = -\frac{1}{\kappa_5^2} \int d^4x \int_0^\pi d\phi r_c |e| (-\Lambda_5 + T) \quad (41)$$

where we have introduced $y = r_c \phi$ and ϕ goes from 0 to 2π . r_c essentially is the compactification radius of the extra dimensional circle. In order to study the effective 4-D action, we begin by considering small fluctuations around the 4-D tetrad. This can be achieved by replacing the Minkowski metric in (20) by a four dimensional metric $\bar{g}_{\mu\nu}(x)$ where

$$\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \bar{h}_{\mu\nu} \quad (42)$$

so we have

$$ds^2 = dy^2 + e^{-2b|y|} (\eta_{\mu\nu} + \bar{h}_{\mu\nu}) dx^\mu dx^\nu \quad (43)$$

In terms of the tetrad field, the 4-D fluctuations can be written as

$$\bar{e}^i_{\mu}(x) = \delta^i_{\mu} + \bar{h}^i_{\mu}(x) \quad (44)$$

where \bar{e}^i_{μ} is the 4-D tetrad.

up to the first order in perturbations, the non-zero torsion components of the tetrad (43) are

$$\begin{aligned} T^0_{50} &= -b(1 + h^0_0 + h^0_{,0}) , \\ T^1_{51} &= -b(1 + h^1_1 + h^1_{,1}) , \quad T^2_{52} = -b(1 + h^2_2 + h^2_{,2}) \\ T^3_{53} &= -b(1 + h^3_3 + h^3_{,3}) \\ T^{\rho}_{5\nu} &= -b(\delta^{\rho}_i h^i_{\nu} + \delta^i_{\nu} h^{\rho}_i) , \quad T^{\rho}_{\mu\nu} = e^{-b|y|} e^{b|y|} \bar{T}^{\rho}_{\mu\nu} = \bar{T}^{\rho}_{\mu\nu} \end{aligned} \quad (45)$$

where $\bar{T}^{\rho}_{\mu\nu}$ is the torsion constructed by the 4 dimensional tetrad (44) . Similarly for the superpotential we have

$$\begin{aligned} S_0^{50} &= \frac{3}{2}b - b(1 + h^0_0 + h^0_{,0}) , \quad S_1^{51} = \frac{3}{2}b(1 + h^1_1 + h^1_{,1}) \\ , \quad S_2^{52} &= \frac{3}{2}b(1 + h^2_2 + h^2_{,2}) , \quad S_3^{53} = \frac{3}{2}b(1 + h^3_3 + h^3_{,3}), \quad S^{\mu\nu}_{\rho} = e^{b|y|} \bar{S}^{\mu\nu}_{\rho} \end{aligned} \quad (46)$$

And finally for the torsion scalar T we have

$$T = -6b^2 \left[1 + 2\text{Tr}(h^i_{\mu}) + 2\text{Tr}(h^{\mu}_i) + \sum_{i=0}^3 (h^i_{\mu} \delta^{\mu}_i)^2 \right] + e^{b|y|} \bar{T} \quad (47)$$

which again \bar{T} is the torsion scalar constructed by (44) and is purely 4 dimensional and y-independent. If we work in the Transverse-Traceless gauge then $\text{Tr}(h^i_{\mu}) = \text{Tr}(h^{\mu}_i) = 0$ and up to first order we have

$$T = -6b^2 + e^{b|y|} \bar{T} . \quad (48)$$

Note that the fourth term in the RHS of (47) is a term second order in fluctuation and can be neglected. For the determinant of pentad we can easily see that there exists the following relation between 5-D determinant constructed by (43) and 4-D determinant constructed by (44)

$$e = e^{-4b|y|} \bar{e} \quad (49)$$

where \bar{e} is y -independent. Substituting (48) and (49) in the action (41), we have

$$S_{grav} = -\frac{1}{\kappa_5^2} \int d^4x \int_0^\pi d\phi r_c e^{-4br_c|\phi|} \bar{e}(-6b^2 + e^{br_c|\phi|} \bar{T}). \quad (50)$$

The first term in the above integral is a constant and can be integrated and absorbed into the cosmological constant term in (41). The second term gives us the effective 4-D action and effective 4-D Planck scale. In this case when we consider only the first order fluctuations, the effective 4-D action is indeed only \bar{T} and as a result this geometrical setup is equivalent to GR in both levels of the action (up to the first order) and the field equations.

4. FRW brane embedded in AdS bulk

For the embedding of a (not necessarily static) maximally 3-symmetric geometry in a 5-dimensional bulk, the most general line element which respects the maximal 3- symmetry is given by [19]

$$ds^2 = -n^2(y, t)dt^2 + 2c(y, t)dydt + a^2(y, t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] + b^2(y, t)dy^2 \quad (51)$$

The most general pentad which gives this geometry and also respect the fundamental structure of spacetime as given by eq (19) is

$$e_M^a = \begin{pmatrix} \sqrt{\left(\frac{c^2}{b^2} + n^2\right)} & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ \frac{c}{b} & 0 & 0 & 0 & b \end{pmatrix}, \quad (52)$$

However starting from TEGR in 5 dimensions and noting that the 5D field equations is invariant under local Lorentz transformation of the pentad,

$$e_M^a \rightarrow \Lambda_b^a e_M^b \quad (53)$$

, one can write most general FRW pentad as without loss of generality

$$e_\mu^A = \begin{pmatrix} n & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \quad (54)$$

The TEGR Lagrangian is also invariant under general coordinate transformation. Using this extra freedom, we choose the gauging $b = 1$ which corresponds to the Gaussian normal gauge in general relativity and effectively fixes the position of the brane in the 5D geometry. For simplicity we choose $y = 0$ as the position of the brane. Note that setting $b = 1$ brings down the number of independent coefficients of the pentad (54) to two. This is acceptable as the number of independent metric coefficients in the corresponding general relativistic setup is also two and we expect our 5D TEGR theory to possess the same number of degrees of freedom as general relativity in five dimensions. The transformation which transforms the pentad (52) to (54) is given by

$$M^A_B = \begin{pmatrix} \frac{n}{\sqrt{(\frac{c^2}{b^2} + n^2)}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{-c}{\sqrt{(\frac{c^2}{b^2} + n^2)}} & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (55)$$

in the $t - y$ plane we have

$$M^A_B = \begin{pmatrix} \frac{n}{\sqrt{(\frac{c^2}{b^2} + n^2)}} & 0 \\ \frac{-c}{\sqrt{(\frac{c^2}{b^2} + n^2)}} & 1 \end{pmatrix}, \quad (56)$$

defining $\cos(\theta) = \frac{\sqrt{(\frac{c^2}{b^2} + n^2)}}{n}$, we see that

$$M^A_B = \begin{pmatrix} \frac{1}{\cos(\theta)} & 0 \\ -\tan(\theta) & 1 \end{pmatrix}, \quad (57)$$

which is exactly the transformation between a non-orthogonal basis and an orthogonal one. Note that this is not a local Lorentz transformation as it does not satisfy the condition $\Lambda^T \eta \Lambda = \eta$ where η is the Minkowski metric of the tangent space.

The fact that these two pentads which are not connected to each other through a local Lorentz transformation, both describe the same geometry with the same dynamics, implies that there are extra hidden symmetries in this setup of the theory that can be used to reduce the number of independent degrees of freedom to that of general relativity in 5 dimensions.

The situation of the effective 4D dynamics on the brane is quite a different matter. As the pentad coefficients in (54) are not separable functions of t and y , finding of the 4D effective dynamics is not a straightforward question like the Minkowski case. The 4D brane dynamics is derived from the bulk quantities through the junction conditions. As the junction conditions in the teleparallel braneworld gravity differ from general relativity, one could expect some modifications in 4 dimensions. The induced field equation derived in [13] shows this feature. Using the FRW pentad (54) with $b = 1$, we derive the torsion, contortion and the superpotential tensors of the 5D background. The torsion scalar then reads

$$T = \frac{-6a'n'}{an} + \frac{6\dot{a}^2}{a^2n^2} - \frac{6a'^2}{a^2} \quad (58)$$

substituting in the 5D teleparallel field equations we get

$$F_{.5}^0 = F_{.0}^5 = -\frac{3}{2} \frac{\dot{a}'}{a^4n^3} + \frac{3}{2} \frac{n'\dot{a}}{a^4n^4} = 0 \quad (59)$$

$$F_{.0}^0 = -3\frac{a'^2}{a^2} - 3\frac{a''}{a} + \frac{3\dot{a}^2}{a^2n^2} = 4\pi G \left(\rho(t)\delta(y) + \Lambda_5 \right) \quad (60)$$

$$F_{.1}^1 = -2a''a + a'^2 + \frac{2aa'n'}{n} + \frac{n''a^2}{n} - \frac{\dot{a}^2}{n^2} + \frac{3a\dot{a}\dot{n}}{n^3} - \frac{2a\ddot{a}}{n^2} = 4\pi G \left(p(t)\delta(y) + \Lambda_5 \right) \quad (61)$$

$$F_{.5}^5 = 3\frac{a'^2}{a^2} + 3\frac{a'n'}{an} - \frac{3\dot{a}^2}{a^2n^2} + \frac{3\dot{a}\dot{n}}{n^3} + 3\frac{\ddot{a}}{a} = 4\pi G\Lambda_5 \quad (62)$$

Here ‘dot’ denotes derivative with respect to time t and a ‘prime’ denotes derivative with respect to y . Note that these are exactly the same 5D equations as in general relativity [20, 21]. This is expected as we are working in the teleparallel equivalent of general relativity in 5 dimensions. Any possible difference in effective 4D theory is then coming from the different junction conditions. To solve these equations, we first solve them in the bulk and then impose the junction conditions at the brane. Equation (59) can be simplified to

$$\frac{\dot{a}'}{a} = \frac{n'}{n} \quad (63)$$

The non-trivial solution to this equation is

$$\dot{a} = D(t)n \quad (64)$$

where $D(t)$ is a function that depends only on time and is independent of y . Substituting this solution in equation (60) and equating it to $-6b^2$ in the bulk gives

$$\frac{D(t)}{a^2} - \frac{a'^2}{a^2} - \frac{a''}{a} = -2b^2 \quad (65)$$

the solution is

$$a(y, t) = \frac{\sqrt{2}}{2} \frac{\sqrt{e^{-2b|y|} (-e^{-2b|y|} D(t) + b e^{-4b|y|} A(t) - b B(t))}}{b e^{-2b|y|}} \quad (66)$$

where $A(t)$ and $B(t)$ are arbitrary t-dependant integration functions. Imposing the junction conditions (15) on the brane we get

$$\frac{3}{2} \frac{a'(0, t)}{a(0, t)} = 4\pi G\rho \quad (67)$$

$$\frac{a'(0, t)}{a(0, t)} + \frac{n'(0, t)}{n(0, t)} = 4\pi Gp \quad (68)$$

Substituting (68) and (69) in (71) and (72) yields

$$\frac{-3b^2 (A(t) + B(t))}{2 A(t)b - 2 B(t)b - 2 D(t)} = 4\pi G\rho(t) \quad (69)$$

$$\begin{aligned} & \frac{\left((B(t) + A(t)) \dot{D}(t) + (-B(t)b - 2 D(t) + 3 A(t)b) \dot{B}(t) \right) b^2}{\left(-D(t) + A(t)b - B(t)b \right) \left(-\dot{D}(t) + \dot{A}(t)b - \dot{B}(t)b \right)} \\ & + \frac{b^2 \left(\dot{A}(t) (-3 B(t)b - 2 D(t) + A(t)b) \right)}{\left(-D(t) + A(t)b - B(t)b \right) \left(-\dot{D}(t) + \dot{A}(t)b - \dot{B}(t)b \right)} \\ & + \frac{-3b^2 (A(t) + B(t))}{2 A(t)b - 2 B(t)b - 2 D(t)} = 4\pi Gp(t) \end{aligned} \quad (70)$$

By assigning appropriate $\rho(t)$ and $p(t)$, these two equations along with the 5-5 equation (62) when evaluated at the brane, will fully specify all tetrad coefficients and with that, the dynamics of a FRW brane embedded in an AdS bulk will be fully determined.

Substituting (58) in the action (41) in the case of a FRW brane, and explicitly evaluating the integral over y , will give us effective 4D action as

$$\begin{aligned} \mathbb{L}_{eff} = & \Phi \arctan \left[\frac{\dot{D} - 2\dot{A}b}{\sqrt{-4b^2\dot{B}\dot{A} - \dot{D}^2}} \right] + \Psi \arctan \left[\frac{D - 2Ab}{\sqrt{4b^2BA - D^2}} \right] \\ & + \Upsilon \ln \left[\frac{D - bA - bB}{\dot{D} - \dot{A}b + \dot{B}b} \right] \end{aligned} \quad (71)$$

where

$$\Phi \equiv \frac{24b(B\dot{A}D + A\dot{B}D - 2A\dot{D}B)(b^2\dot{B}\dot{A} - \frac{1}{4}D^2)}{b^2\dot{A}^3B^2 - 2b^2AB\dot{B}\dot{A} - D^2\dot{B}\dot{A} + D\dot{D}BA} \quad (72)$$

$$\begin{aligned} \Psi \equiv & 24b \left\{ \frac{-1}{2}b^2A^2D^2B^2 + \frac{1}{2}D^4A\dot{B} - \frac{1}{2}D^3A\dot{B} \right. \\ & + b^2BD^2A^2\dot{B} - 2b^2A^3DB\dot{B} + b^2A^2B^2\dot{A}\dot{D} \\ & - \frac{1}{2}ABD^3\dot{A}\dot{D} + \frac{1}{4}ABD^2\dot{A}\dot{D} - \frac{1}{2}b^2A^2D^2\dot{B}^3\dot{D} \\ & \left. + b^2A^2B\dot{B}\dot{D} - \frac{1}{2}AD^3\dot{B}\dot{D} + \frac{1}{4}AD^2\dot{B}\dot{D} + \frac{1}{2}ABD^2\dot{D}^2 \right\} \\ & \times \left\{ b^2\dot{A}^3B^2 - 2b^2AB\dot{B}\dot{A} - D^2\dot{B}\dot{A} + D\dot{D}BA \right\}^{-1} \end{aligned} \quad (73)$$

$$\Upsilon \equiv \frac{24b \left[\frac{1}{4}(b^2A\dot{B} - \frac{1}{2}D\dot{D} + b^2AB)(A\dot{B} - B\dot{A}) \right]}{b^2\dot{A}^3B^2 - 2b^2AB\dot{B}\dot{A} - D^2\dot{B}\dot{A} + D\dot{D}BA} \quad (74)$$

As we can see, the effective Lagrangian in the FRW case is not the same as in GR and as a result the 4-D cosmological dynamics will be different from GR. For a quick glance at the practical results of the model and study quantitative differences, we consider a simple inflationary universe when the exponential expansion is driven by a scalar field. Using the procedure above and eq (16), the Friedmann equation on the brane will be [14]

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3} \left[-\Lambda_4 + 8\pi G\rho + \frac{1}{4}(4\pi G)^2(11 - 60\omega + 93\omega^2)\rho^2 \right] \quad (75)$$

where $\omega = \frac{p}{\rho}$ is the equation of state parameter of the matter confined to the brane. The scalar field, ϕ which drives the inflation has energy density and pressure

$$\rho = \frac{1}{2}\dot{\phi}^2 + V \quad , \quad p = \frac{1}{2}\dot{\phi}^2 - V \quad (76)$$

respectively where $V(\phi)$ is the inflation potential. We define the slow-roll parameters as usual

$$\epsilon \equiv \frac{M_4^2}{4\pi} \left(\frac{H'}{H} \right)^2 \quad , \quad \eta \equiv \frac{M_4^2}{4\pi} \left(\frac{H''}{H} \right) \quad (77)$$

In the slow-roll regime we have

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \quad , \quad 3H\dot{\phi} \simeq -V'(\phi) \quad (78)$$

We assume a chaotic type potential for the inflaton field

$$V(\phi) = \frac{1}{2}m^2\phi^2 \quad (79)$$

Substituting we have

$$\epsilon = \pi G \left(\frac{V'}{V} \right)^2 \left(\frac{1 + 1312\pi G V}{1 + 656\pi G V} \right)^2 \quad (80)$$

$$\eta = \pi G \left(\frac{V''}{V} \right) \left[\frac{26896(4\pi G)^2 V^2 + 246(4\pi G)V + 8\pi G - \frac{V'^2}{V''V}}{(1 + (4\pi G)164V)^2} \right] \quad (81)$$

where here a prime denotes the differentiation with respect to the argument. Comparing these relations with the corresponding results in general relativity in ref [22], we find that contrary to GR case, here the slow-roll parameters are enhanced by brane modifications.

Other inflation parameters can then be derived using standard procedures. The scalar spectral index will be

$$\begin{aligned} n_s = 1 - 6\pi G \left(\frac{V'}{V} \right)^2 \left(\frac{1 + 1312\pi G V}{1 + 656\pi G V} \right)^2 \\ + 2\pi G \left(\frac{V''}{V} \right) \left[\frac{26896(4\pi G)^2 V^2 + 246(4\pi G)V + 8\pi G - \frac{V'^2}{V''V}}{(1 + (4\pi G)164V)^2} \right] \end{aligned} \quad (82)$$

The amplitude of scalar and tensor perturbations then are

$$A_S^2 = \frac{9}{25} \frac{\left[\frac{8}{3} \pi G V + \frac{656}{3} \pi^2 G^2 V^2 \right]^6}{V'^2} \quad (83)$$

and

$$A_T^2 = \frac{1}{1600} \frac{\left[\frac{8}{3} \pi G V + \frac{656}{3} \pi^2 G^2 V^2 \right]^2}{\pi^5 G^4} \quad (84)$$

Respectively, then the tensor-to-scalar ratio will be

$$\frac{A_T^2}{A_S^2} = \frac{1}{576} \frac{V'^2}{\left[\frac{8}{3} \pi G V + \frac{656}{3} \pi^2 G^2 V^2 \right]^4 \pi^5 G^4} \quad (85)$$

5. Conclusion and discussion

Teleparallel gravity as the gauge theory for the translation group, offers a viable gravitational theory for macroscopic matter. There exist one class of teleparallel Lagrangians, called teleparallel equivalent of general relativity (TEGR), which for all practical purposes is empirically indistinguishable from general relativity for scalar matter and electromagnetic fields. For finding possible observational differences between TG and GR, we considered a 5 dimensional braneworld setup. The presence of the brane as a boundary hypersurface embedded in the bulk, where all the ordinary matter fields are confined to the brane and only gravitons can propagate in the fifth dimension, offers an interesting opportunity to study possible differences between TG and GR. In this paper, using the results of ref [13], we constructed a RS-type braneworld model in a teleparallel background. Starting from TEGR in 5 dimensions, we investigated possible local Lorentz invariance violations in the effective 4 dimensional theory. In both cases of Minkowski and FRW branes, the 4-D effective field equations found to be local Lorentz invariant. Any possible difference between TG and GR in the effective 4D dynamics, is a result of different junction conditions in these two theories. In TG setup, the second junction condition relates the jump in the superpotential tensor across the brane to the matter content confined to the brane. This is in stark difference to GR where the second junction condition involves the extrinsic curvature. For the case of a FRW brane embedded in AdS bulk, we

studied both the background dynamics. FRW pentad coefficients have been derived using the 5D field equations and teleparallel junction conditions. The bulk field equations are exactly the same as GR, however deriving the effective 4D equations involves matching the discontinuities on both sides of the teleparallel field equations via the junction conditions. As a result of different junction conditions, the 4-D cosmological evolution will be different in teleparallel gravity compared to GR. For a quick illustration of practical results, we considered a simple inflationary scenario where the 4D exponential expansion is driven by a scalar field. In the slow-roll regime we found that the slow-roll parameters are enhanced by braneworld modifications in teleparallel gravity. This is quite different to the general relativistic results in [22] where the slow-roll parameters were suppressed. This means that for a given potential, the inflation will end sooner in teleparallel gravity than in general relativity.

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